A CBER $\quad X$: a standard Borel sp.

<u>Def</u> (X, E) is a CBER

$\iff$ · $E \subset X \times X$ Borel

· E is equiv. rel on X

· $[x]_E$ is countable $\forall x \in X$

<u>e.g.</u> A countable grp $\Gamma \overset{\alpha}{\curvearrowright} X$

$$E_\alpha = \{(x, \gamma x) \mid x \in X, \gamma \in \Gamma\}$$

(X, G) is a Borel graph

$\Leftrightarrow$ · $G \subset X \times X \setminus \Delta_X$

· $(x, y) \in G \Rightarrow (y, x) \in G$

If G is loc. fin

$\rightsquigarrow$ G generates a

CBER E_G

e.g. $\Gamma = \langle \Sigma \rangle$ a f.g grp $\overset{\alpha}{\curvearrowright} X$

$Sch(\alpha, \Sigma) = \{ (x, \gamma x) \mid x \in X \ \gamma \in \Sigma^{\pm 1} \}$

Def A CBER is treeable if it is generated by a Borel acyclic graph.

e.g. $F_2 = \langle a, b \rangle \overset{\alpha}{\curvearrowright} X$ freely

$Sch(\alpha, \{a, b\})$ is acyclic

Jackson-Kechris-Louveau '02

TFAE · E is treeable

· $\exists\, F_2 \overset{\alpha}{\curvearrowright} Y$ free s.t. $(X, E) \leq_B (Y, E_\alpha)$

- Virtually free grps

Fact Γ f.g. grp TFAE

- Γ is virtually free
- Cayley graph of Γ is QI to a tree
- $cd_{\mathbb{Q}}(\Gamma) \leq 1$ (Dunwoody '79)

$H^k(\Gamma, M) = 0 \quad \forall k > 1$

$\forall M$ left $\mathbb{Q}\Gamma$-module

JKL'02 Orbit equiv. rel of free action of virtually free grps are treeable.

Chen–Poulin–Tao–Tserunyan '25

(X, G) loc. fin.

If every component of G is QI to a tree

then E_G is treeable.

i.e. $\exists T$ Borel acyclic $E_G = E_T$.

Q. Is T "close" to G?

CPTT If G has UBD, then Yes.

$(X,G) \underset{\mathrm{Lip}}{\sim} (X,T) \iff \exists L \geq 1 \quad L^{-1} d_G \leq d_T \leq L d_G$

- Main theorem

<u>Thm</u> (I.) $\quad (X,G)$ UBD $\quad R \neq 0$ comm. ring

TFAE

(1) $\exists T$ a Borel acyclic graph on X s.t. $T \overset{\mathrm{Lip}}{\sim} G$

(2) $\quad cd_R(G) \leq 1$

(X,G) an abstract graph UBD

$\bigoplus \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$

Σ

$\delta_s \mapsto \delta_s - \delta_e$

$$G^n = \{(x,y) \mid d_G(x,y) \leq n\}$$

$\cap$

X^2

$$\mathbb{Z}_G = \{a : X^2 \to \mathbb{Z} \text{ bounded fct} \mid \exists n \ \operatorname{supp}(a) \subset G^n\}$$

$$(a \cdot b)(x,y) = \sum_{z \in X} a(x,z)\, b(z,y) \qquad a, b \in \mathbb{Z}_G$$

$\rightsquigarrow$ a ring with unit 1_{Δ_X}

$\ell^\infty(X) = \{ X \to \mathbb{Z} \text{ bounded} \}$

is a left $\mathbb{Z}_G$-module

$$(af)(x) = \sum_{y \in X} a(x,y) f(y) \qquad a \in \mathbb{Z}_G, \ f \in \ell^\infty(X)$$

<u>Def</u> For a left $\mathbb{Z}_G$-module M

$$H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}_G}(\ell^\infty(X), M).$$

① Take a projective resolution of $\ell^\infty(X)$ over $\mathbb{Z}_G$

i.e. $\xrightarrow{\partial_n} P_n \to \cdots P_1 \xrightarrow{\partial_0} \mathbb{Z}_G \xrightarrow{\varepsilon} \ell^\infty(X) \to 0$

$1_{\Delta_X} \mapsto 1_X$

exact, all P_i projective

② Apply $\mathrm{Hom}_{\mathbb{Z}_G}(-, M)$

$\xleftarrow{\partial_n^*} \mathrm{Hom}_{\mathbb{Z}_G}(P_n, M) \leftarrow \cdots \xleftarrow{\partial_0^*} \mathrm{Hom}_{\mathbb{Z}_G}(P_0, M)$

$\varphi \circ \partial_n \longleftarrow\!\!\shortmid \quad \overset{\Psi}{\varphi}$

③ $H^n(G, M) = \begin{cases} \ker \partial_n^* / \mathrm{im}\, \partial_{n-1}^* & (n \geq 1) \\ \ker \partial_0^* & (n = 0). \end{cases}$

Def $cd_R(G) = \inf\{n \geq 0 \mid H^k(G, M) = 0 \ \forall k > n,\ \forall M \text{ left } R_G\text{-module}\}$

$R_G = \mathbb{Z}_G \otimes_{\mathbb{Z}} R$

Prop (1) $G \underset{Lip}{\sim} G' \Leftrightarrow \mathbb{Z}_G = \mathbb{Z}_{G'} \Rightarrow cd_R(G) = cd_R(G')$

(2) G acyclic $\Rightarrow cd_R(G) \leq 1$

(X, G) Borel UBD every component QI to a tree w/ uniform const.

$\exists G'$ acyclic $\underset{Lip}{\sim} G$ $\quad cd(G) \leq 1 \rightsquigarrow \exists$ Borel acyclic $\underset{Lip}{\sim} G$. $\square$